

Remarks on dimensions of Cartesian product sets *

Chun WEI, Shengyou WEN[†], Zhixiong WEN

June 18, 2015

Abstract

Given metric spaces E and F , it is well known that

$$\dim_H E + \dim_H F \leq \dim_H(E \times F) \leq \dim_H E + \dim_P F,$$

$$\dim_H E + \dim_P F \leq \dim_P(E \times F) \leq \dim_P E + \dim_P F,$$

and

$$\underline{\dim}_B E + \overline{\dim}_B F \leq \overline{\dim}_B(E \times F) \leq \overline{\dim}_B E + \overline{\dim}_B F,$$

where $\dim_H E$, $\dim_P E$, $\underline{\dim}_B E$, $\overline{\dim}_B E$ denote the Hausdorff, packing, lower box-counting, and upper box-counting dimension of E , respectively. In this note we shall provide examples of compact sets showing that the dimension of the product $E \times F$ may attain any of the values permitted by the above inequalities. The proof will be based on a study on dimension of the product of sets defined by digit restrictions.

Key Words Hausdorff dimension, packing dimension, box-counting dimension, Cartesian products

2010 MSC 28A80, 11K55

1 Introduction

For the dimensions of Cartesian products, it is well known that

$$\dim_H E + \dim_H F \leq \dim_H(E \times F) \leq \dim_H E + \dim_P F, \quad (1)$$

where $E, F \subset \mathbb{R}^d$. Hereafter $\dim_H E$ and $\dim_P E$ denote the Hausdorff and packing dimension of E , respectively. With some additional hypotheses, the left hand inequality was first obtained by Besicovitch and Moran [1]. Marstrand [7] proved it without these hypotheses. The right hand side is due to Tricot [10]. He also proved that

$$\dim_H E + \dim_P F \leq \dim_P(E \times F) \leq \dim_P E + \dim_P F. \quad (2)$$

*Supported by the NSFC (Nos. 11271114, 11301162, 11371156, 11431007).

[†]Corresponding author: sywen_65@163.com

Howroyd [5] proved that formulas (1) and (2) are still valid for arbitrary metric spaces. For the upper box-counting dimension of the product $E \times F$ one has

$$\underline{\dim}_B E + \overline{\dim}_B F \leq \overline{\dim}_B(E \times F) \leq \overline{\dim}_B E + \overline{\dim}_B F, \quad (3)$$

where $\underline{\dim}_B E$ and $\overline{\dim}_B E$ denote the lower and upper box-counting dimension of E , respectively; see [11]. For the product $X := \prod_{i=1}^d (X_i, \rho_i)$ of metric spaces (X_i, ρ_i) we always assume that it has been equipped with the metric

$$\rho(x, y) = \left(\sum_{i=1}^d (\rho_i(x_i, y_i))^2 \right)^{\frac{1}{2}}, \quad x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in X.$$

In the present paper we shall provide examples of compact sets showing that the dimension of the product $E \times F$ may attain any of the values permitted by the above inequalities. Our main results are the following three theorems.

Theorem 1. *Let $\alpha, \beta, \gamma, \lambda$ be positive with $\beta \leq \gamma$ and $\alpha + \beta \leq \lambda \leq \alpha + \gamma$. Then there are compact metric spaces E, F such that*

$$\dim_H E = \alpha, \dim_H F = \beta, \dim_P F = \gamma, \dim_H(E \times F) = \lambda.$$

Theorem 2. *Let $\alpha, \beta, \gamma, \lambda$ be positive with $\alpha \leq \gamma$ and $\alpha + \beta \leq \lambda \leq \gamma + \beta$. Then there are compact metric spaces E, F such that*

$$\dim_H E = \alpha, \dim_P F = \beta, \dim_P E = \gamma, \dim_P(E \times F) = \lambda.$$

Theorem 3. *Let $\alpha, \beta, \gamma, \lambda$ be positive with $\alpha \leq \gamma$ and $\alpha + \beta \leq \lambda \leq \gamma + \beta$. Then there are compact metric spaces E, F such that*

$$\underline{\dim}_B E = \alpha, \overline{\dim}_B F = \beta, \overline{\dim}_B E = \gamma, \overline{\dim}_B(E \times F) = \lambda.$$

For Assouad dimension it is known that

$$\max\{\dim_A E, \dim_A F\} \leq \dim_A(E \times F) \leq \dim_A E + \dim_A F$$

for arbitrary metric spaces E and F , where $\dim_A E$ denotes the Assouad dimension of E ; see J. Luukkainen [6]. Based on a study on the Assouad dimension of uniform Cantor sets, Peng-Wang-Wen [9] proved that the Assouad dimension of the product $E \times F$ may attain any of the values permitted by this inequality.

The proof of the main results of this paper will be based on a study on dimension of Cartesian products of sets defined by digit restrictions.

2 Proofs of main results

We begin with a study on dimension of products of sets defined by digit restrictions.

Let \mathbb{N} be the set of positive integers and S a nonempty proper subset of \mathbb{N} . Denote by $d_k(S)$ the density of S in $\{1, 2, \dots, k\}$, i.e.

$$d_k(S) = \frac{\sharp(S \cap \{1, 2, \dots, k\})}{k},$$

where $\sharp A$ denotes the number of elements of the set A . We call

$$\overline{d}(S) = \limsup_{k \rightarrow \infty} d_k(S) \quad \text{and} \quad \underline{d}(S) = \liminf_{k \rightarrow \infty} d_k(S) \quad (4)$$

the upper and lower density of S in \mathbb{N} , respectively. Consider the binary expansion of numbers in $[0, 1]$ and define a subset of $[0, 1]$ by

$$E_S := \left\{ \sum_{k \in \mathbb{N}} \frac{a_k}{2^k} : a_k = 0 \text{ for all } k \notin S \right\}. \quad (5)$$

We shall construct sets of this type to prove our results. First of all, we have from ([3], p.12, 13, 20, 21)

$$\dim_H E_S = \underline{\dim}_B E_S = \underline{d}(S) \quad \text{and} \quad \overline{\dim}_B E_S = \overline{d}(S). \quad (6)$$

On the other hand, since the set E_S has the property that

$$\overline{\dim}_B(E_S \cap V) = \overline{\dim}_B E_S \quad (7)$$

for all open sets V that intersect E_S , one has from ([4], Corollary 3.9)

$$\dim_P E_S = \overline{\dim}_B E_S = \overline{d}(S). \quad (8)$$

Let S_1, S_2, \dots, S_d be nonempty proper subsets of \mathbb{N} and $E_{S_1}, E_{S_2}, \dots, E_{S_d}$ the corresponding subsets of $[0, 1]$ defined by (5). For our purpose we shall study the dimension of the cartesian product $\prod_{i=1}^d E_{S_i}$ in the following. For simplicity we shall write E_S^d for $\prod_{i=1}^d E_{S_i}$ when $S_i = S$ for all i .

Lemma 1. *Let S, S_1, S_2, \dots, S_d be nonempty proper subsets of \mathbb{N} . Then*

$$\dim_H \prod_{i=1}^d E_{S_i} = \underline{\dim}_B \prod_{i=1}^d E_{S_i} = \liminf_{k \rightarrow \infty} \sum_{i=1}^d d_k(S_i) \quad (9)$$

and

$$\dim_P \prod_{i=1}^d E_{S_i} = \overline{\dim}_B \prod_{i=1}^d E_{S_i} = \limsup_{k \rightarrow \infty} \sum_{i=1}^d d_k(S_i). \quad (10)$$

In particular, when $S_i = S$ for all i we have

$$\dim_H E_S^d = \underline{\dim}_B E_S^d = d \underline{d}(S) \quad (11)$$

and

$$\dim_P E_S^d = \overline{\dim}_B E_S^d = d \overline{d}(S). \quad (12)$$

Proof. For each $x \in [0, 1]^d$ and each integer k let $I_k(x)$ denote the unique k -level dyadic cube of the form

$$\prod_{i=1}^d [\frac{j_i - 1}{2^k}, \frac{j_i}{2^k})$$

containing x . Then, given k , the family of k -level dyadic cubes that intersect $\prod_{i=1}^d E_{S_i}$ is

$$\{I_k(x) : x \in \prod_{i=1}^d E_{S_i}\}.$$

From the definition of the set $\prod_{i=1}^d E_{S_i}$ this family is of cardinality

$$\#\{I_k(x) : x \in \prod_{i=1}^d E_{S_i}\} = \prod_{i=1}^d 2^{\#(S_i \cap \{1, 2, \dots, k\})} = 2^{k \sum_{i=1}^d d_k(S_i)}.$$

It follows from the definition of box-counting dimension that

$$\overline{\dim}_B \prod_{i=1}^d E_{S_i} = \limsup_{k \rightarrow \infty} \sum_{i=1}^d d_k(S_i) \quad \text{and} \quad \underline{\dim}_B \prod_{i=1}^d E_{S_i} = \liminf_{k \rightarrow \infty} \sum_{i=1}^d d_k(S_i). \quad (13)$$

Observing that $\prod_{i=1}^d E_{S_i}$ has the homogeneity as in (7), we get

$$\dim_P \prod_{i=1}^d E_{S_i} = \overline{\dim}_B \prod_{i=1}^d E_{S_i}.$$

This completes the proof of (10).

Now we prove the equality (9). Let μ be the unique Borel probability measure on $\prod_{i=1}^d E_{S_i}$ such that

$$\mu(I_k(x)) = 2^{-k \sum_{i=1}^d d_k(S_i)}$$

for any $x \in \prod_{i=1}^d E_i$. It follows that

$$\liminf_{k \rightarrow \infty} \frac{\log \mu(I_k(x))}{\log |I_k(x)|} = \liminf_{k \rightarrow \infty} \sum_{i=1}^d d_k(S_i),$$

where $|I_k(x)|$ denotes the diameter of $I_k(x)$. Then we get from Billingsley's lemma ([3], Lemma 3.1) that

$$\dim_H \prod_{i=1}^d E_{S_i} = \liminf_{k \rightarrow \infty} \sum_{i=1}^d d_k(S_i),$$

which, combined with (13), gives (9). \square

Lemma 2. *Let S and T be nonempty proper subsets of \mathbb{N} , E_S and E_T be the corresponding subsets of $[0, 1]$ defined by (5), and $d \geq 1$ be an integer. Then*

$$\dim_H(E_S \times E_T)^d = d \dim_H(E_S \times E_T).$$

A similar equality holds for both the packing dimension and the upper box-counting dimension.

Proof. It is immediate by Lemma 1. \square

We remark that the equality $\dim_H E^d = d \dim_H E$ is not true in general. A counterexample can be found in Remark 1 at the end of this section.

As mentioned, we shall construct sets of the form E_S to prove our theorems. The set E_S is determined by the digit set S that will be chosen as follows: Let $a_1, a_2 \in (0, 1)$ be fixed. Let $\{k_n\}_{n \geq 0}$ be a sequence of positive integers such that

$$\lim_{n \rightarrow \infty} \frac{k_n}{k_{n+1}} = 0 \quad (14)$$

and

$$(k_{n+1} - k_n) \min\{a_1, a_2\} > 1 \quad (15)$$

for all n . For each $j \geq 0$ and $i \in \{1, 2\}$ let M_{2j+i} be the smallest integer bigger than or equal to $a_i(k_{2j+i} - k_{2j+i-1})$, in other words, M_{2j+i} is an integer such that

$$M_{2j+i} - 1 < a_i(k_{2j+i} - k_{2j+i-1}) \leq M_{2j+i}.$$

Then one has

$$\left\lceil \frac{M_{2j+i} - 1}{a_i} \right\rceil < k_{2j+i} - k_{2j+i-1} \leq \left\lceil \frac{M_{2j+i}}{a_i} \right\rceil,$$

where $\lceil x \rceil$ denotes the biggest integer smaller than or equal to x . Let

$$A_{ji} = \left\{ k_{2j+i-1} + \left\lceil \frac{1}{a_i} \right\rceil, k_{2j+i-1} + \left\lceil \frac{2}{a_i} \right\rceil, \dots, k_{2j+i-1} + \left\lceil \frac{M_{2j+i} - 1}{a_i} \right\rceil, k_{2j+i} \right\}$$

and let $A_j = A_{j1} \cup A_{j2}$. Then A_j is a subset of $\{k_{2j} + 1, \dots, k_{2j+2}\}$. We define a subset of \mathbb{N} by

$$S(\{k_n\}_{n \geq 0}, a_1, a_2) := \bigcup_{j=0}^{\infty} A_j. \quad (16)$$

In what follows for a subset of \mathbb{N} of this type we always assume that the related defining data satisfy (14) and (15).

Lemma 3. *Let $S = S(\{k_n\}_{n \geq 0}, a_1, a_2)$ be a subset of \mathbb{N} defined by (16), and E_S the corresponding subset of $[0, 1]$ defined by (5). Then*

$$\dim_H E_S = \underline{\dim}_B E_S = \min\{a_1, a_2\} \quad (17)$$

and

$$\dim_P E_S = \overline{\dim}_B E_S = \max\{a_1, a_2\}. \quad (18)$$

Proof. From Lemma 1 we only need show that

$$\underline{d}(S) = \min\{a_1, a_2\} \quad \text{and} \quad \overline{d}(S) = \max\{a_1, a_2\}. \quad (19)$$

For this, we are going to estimate the density $d_k(S)$ of S in $\{1, 2, \dots, k\}$. First of all, given $j \geq 0$ and $i \in \{1, 2\}$, we have from the definition of $\{M_k\}_{k=1}^\infty$

$$\sum_{q=1}^{2j+i} M_q = \sum_{q=1}^{2j+i-1} M_q + M_{2j+i} \leq k_{2j+i-1} + a_i(k_{2j+i} - k_{2j+i-1}) + 1$$

and

$$\sum_{q=1}^{2j+i} M_q \geq M_{2j+i} \geq a_i(k_{2j+i} - k_{2j+i-1}),$$

so it follows from (14) that

$$\lim_{j \rightarrow \infty} d_{k_{2j+i}}(S) = \lim_{j \rightarrow \infty} \frac{\sum_{q=1}^{2j+i} M_q}{k_{2j+i}} = a_i, \quad i = 1, 2. \quad (20)$$

To estimate $d_k(S)$ for a general integer $k \in \mathbb{N}$, we let j be an integer such that $k_{2j} \leq k < k_{2j+2}$ and consider two cases as follows.

Case 1. $k_{2j} \leq k < k_{2j+1}$.

In this case, one has an integer $m \in [0, M_{2j+1} - 1]$ such that

$$k_{2j} + \lceil \frac{m}{a_1} \rceil \leq k < k_{2j} + \lceil \frac{m+1}{a_1} \rceil. \quad (21)$$

Then by the definition of the set S one has

$$\#(S \cap \{1, 2, \dots, k\}) = \sum_{q=1}^{2j} M_q + m \leq k_{2j-1} + a_2(k_{2j} - k_{2j-1}) + 1 + m$$

and

$$\#(S \cap \{1, 2, \dots, k\}) = \sum_{q=1}^{2j} M_q + m \geq a_2(k_{2j} - k_{2j-1}) + m.$$

It then follows from (21) that the density $d_k(S)$ satisfies

$$f_j(m) \leq d_k(S) \leq g_j(m), \quad (22)$$

where

$$f_j(m) = a_1 \frac{a_2(k_{2j} - k_{2j-1}) + m}{a_1 k_{2j} + 1 + m}$$

and

$$g_j(m) = a_1 \frac{k_{2j-1} + a_2(k_{2j} - k_{2j-1}) + 1 + m}{a_1 k_{2j} - a_1 + m}.$$

For sufficiently large k we have from (22) the following claims:

- when $a_1 < a_2$, both $f_j(m)$ and $g_j(m)$ are decreasing as m goes from 0 to $M_{2j+1} - 1$, so $f_j(M_{2j+1} - 1) \leq d_k(S) \leq g_j(0)$;
- when $a_1 = a_2$, $f_j(m)$ is increasing and $g_j(m)$ is decreasing as m goes from 0 to $M_{2j+1} - 1$, so $f_j(0) \leq d_k(S) \leq g_j(0)$;
- when $a_1 > a_2$, both $f_j(m)$ and $g_j(m)$ are increasing as m goes from 0 to $M_{2j+1} - 1$, so $f_j(0) \leq d_k(S) \leq g_j(M_{2j+1} - 1)$.

Case 2. $k_{2j+1} \leq k < k_{2j+2}$.

In this case, there is an integer $m \in [0, M_{2j+2} - 1]$ such that

$$k_{2j+1} + \lfloor \frac{m}{a_2} \rfloor \leq k < k_{2j+1} + \lfloor \frac{m+1}{a_2} \rfloor.$$

Then one has

$$\sharp(S \cap \{1, 2, \dots, k\}) = \sum_{q=1}^{2j+1} M_q + m \leq k_{2j} + a_1(k_{2j+1} - k_{2j}) + 1 + m$$

and

$$\sharp(S \cap \{1, 2, \dots, k\}) = \sum_{q=1}^{2j+1} M_q + m \geq a_1(k_{2j+1} - k_{2j}) + m.$$

It then follows that the density $d_k(S)$ satisfies

$$\tilde{f}_j(m) \leq d_k(S) \leq \tilde{g}_j(m), \quad (23)$$

where

$$\tilde{f}_j(m) = a_2 \frac{a_1(k_{2j+1} - k_{2j}) + m}{a_2 k_{2j+1} + 1 + m}.$$

and

$$\tilde{g}_j(m) = a_2 \frac{k_{2j} + a_1(k_{2j+1} - k_{2j}) + 1 + m}{a_2 k_{2j+1} - a_2 + m}.$$

For sufficiently large k we have from (23) the following claims:

- when $a_1 < a_2$, both $\tilde{f}_j(m)$ and $\tilde{g}_j(m)$ are increasing as m goes from 0 to $M_{2j+2} - 1$, so $\tilde{f}_j(0) \leq d_k(S) \leq \tilde{g}_j(M_{2j+2} - 1)$;
- when $a_1 = a_2$, $\tilde{f}_j(m)$ is increasing and $\tilde{g}_j(m)$ is decreasing as m goes from 0 to $M_{2j+2} - 1$, so $\tilde{f}_j(0) \leq d_k(S) \leq \tilde{g}_j(0)$;
- when $a_1 > a_2$, both $\tilde{f}_j(m)$ and $\tilde{g}_j(m)$ are decreasing as m goes from 0 to $M_{2j+2} - 1$, so $\tilde{f}_j(M_{2j+2} - 1) \leq d_k(S) \leq \tilde{g}_j(0)$.

Now we obtain an estimate of the density $d_k(S)$ for each of all kinds of cases. Noting from (14) and the definition of M_{2j+i} that

$$\lim_{j \rightarrow \infty} f_j(M_{2j+1} - 1) = \lim_{j \rightarrow \infty} g_j(M_{2j+1} - 1) = \lim_{j \rightarrow \infty} \tilde{f}_j(0) = \lim_{j \rightarrow \infty} \tilde{g}_j(0) = a_1$$

and

$$\lim_{j \rightarrow \infty} f_j(0) = \lim_{j \rightarrow \infty} g_j(0) = \lim_{j \rightarrow \infty} \tilde{f}_j(M_{2j+2} - 1) = \lim_{j \rightarrow \infty} \tilde{g}_j(M_{2j+2} - 1) = a_2,$$

we get by the above estimates of the density $d_k(S)$

$$\min\{a_1, a_2\} \leq \underline{d}(S) \leq \overline{d}(S) \leq \max\{a_1, a_2\},$$

which, combined with (20), yields (19) as desired. \square

Lemma 4. *Let $S = S(\{k_n\}_{n \geq 0}, a_1, a_2)$ and $T = S(\{k_n\}_{n \geq 0}, b_1, b_2)$ be subsets of \mathbb{N} defined by (16). Let E_S and E_T be the corresponding subsets of $[0, 1]$ defined by (5). Then*

$$\dim_H(E_S \times E_T) = \underline{\dim}_B(E_S \times E_T) = \min\{a_1 + b_1, a_2 + b_2\} \quad (24)$$

and

$$\dim_P(E_S \times E_T) = \overline{\dim}_B(E_S \times E_T) = \max\{a_1 + b_1, a_2 + b_2\}. \quad (25)$$

Proof. From Lemma 1 we only need show that

$$\liminf_{k \rightarrow \infty} (d_k(S) + d_k(T)) = \min\{a_1 + b_1, a_2 + b_2\}$$

and

$$\limsup_{k \rightarrow \infty} (d_k(S) + d_k(T)) = \max\{a_1 + b_1, a_2 + b_2\}.$$

As the sequence $\{M_q\}_{q \in \mathbb{N}}$ in the definition of S , we have a sequence of integers in the definition of T , denoted by $\{N_q\}_{q \in \mathbb{N}}$. Then, for each $j \geq 0$ and $i \in \{1, 2\}$, N_{2j+i} is the smallest integer bigger than or equal to $b_i(k_{2j+i} - k_{2j+i-1})$. As was shown for S in Lemma 3, we have for T

$$\lim_{j \rightarrow \infty} d_{k_{2j+i}}(T) = b_i, \quad i \in \{1, 2\},$$

so

$$\lim_{j \rightarrow \infty} (d_{k_{2j+i}}(S) + d_{k_{2j+i}}(T)) = a_i + b_i, \quad i \in \{1, 2\}. \quad (26)$$

Now, given $k \in \mathbb{N}$, we are going to estimate $d_k(S) + d_k(T)$. Let j be an integer such that $k_{2j} \leq k < k_{2j+2}$. We consider two cases as follows.

Case 1. $k_{2j} \leq k < k_{2j+1}$.

Let $m \in [0, M_{2j+1} - 1]$, $n \in [0, N_{2j+1} - 1]$ be integers such that

$$k_{2j} + \left\lceil \frac{m}{a_1} \right\rceil \leq k < k_{2j} + \left\lceil \frac{m+1}{a_1} \right\rceil,$$

$$k_{2j} + \left\lceil \frac{n}{b_1} \right\rceil \leq k < k_{2j} + \left\lceil \frac{n+1}{b_1} \right\rceil.$$

Then m, n have the following relationship

$$\frac{b_1}{a_1}m - b_1 - 1 \leq n \leq \frac{b_1}{a_1}m + \frac{b_1}{a_1} + b_1. \quad (27)$$

Like the estimate (22) of $d_k(S)$, for the density of T in $\{1, 2, \dots, k\}$ we have

$$f_j^*(m, n) \leq d_k(T) \leq g_j^*(m, n),$$

where

$$f_j^*(m, n) = a_1 \frac{b_2(k_{2j} - k_{2j-1}) + n}{a_1 k_{2j} + 1 + m}$$

and

$$g_j^*(m, n) = a_1 \frac{k_{2j-1} + b_2(k_{2j} - k_{2j-1}) + 1 + n}{a_1 k_{2j} - a_1 + m}.$$

This estimate of $d_k(T)$ together with (22) gives

$$f_j(m) + f_j^*(m, n) \leq d_k(S) + d_k(T) \leq g_j(m) + g_j^*(m, n),$$

which, combined with (27), yields

$$F_j(m) \leq d_k(S) + d_k(T) \leq G_j(m), \quad (28)$$

where

$$F_j(m) = (a_1 + b_1) \frac{\frac{a_1(a_2+b_2)}{a_1+b_1}(k_{2j} - k_{2j-1}) - \frac{a_1+a_1b_1}{a_1+b_1} + m}{a_1 k_{2j} + 1 + m}$$

and

$$G_j(m) = (a_1 + b_1) \frac{\frac{2a_1}{a_1+b_1}k_{2j-1} + \frac{a_1(a_2+b_2)}{a_1+b_1}(k_{2j} - k_{2j-1}) + \frac{b_1+a_1b_1+2a_1}{a_1+b_1} + m}{a_1 k_{2j} - a_1 + m}.$$

For sufficiently large k we have from (28) the following claims:

- when $a_1 + b_1 < a_2 + b_2$, both $F_j(m)$ and $G_j(m)$ are decreasing as m goes from 0 to $M_{2j+1} - 1$, so $F_j(M_{2j+1} - 1) \leq d_k(S) + d_k(T) \leq G_j(0)$;
- when $a_1 + b_1 = a_2 + b_2$, $F_j(m)$ is increasing and $G_j(m)$ is decreasing as m goes from 0 to $M_{2j+1} - 1$, so $F_j(0) \leq d_k(S) + d_k(T) \leq G_j(0)$;
- when $a_1 + b_1 > a_2 + b_2$, both $f_j(m)$ and $g_j(m)$ are increasing as m goes from 0 to $M_{2j+1} - 1$, so $F_j(0) \leq d_k(S) + d_k(T) \leq G_j(M_{2j+1} - 1)$.

Case 2. $k_{2j+1} \leq k < k_{2j+2}$.

Let $m \in [0, M_{2j+2})$, $n \in [0, N_{2j+2})$ be integers such that

$$k_{2j+1} + \left\lceil \frac{m}{a_2} \right\rceil \leq k < k_{2j+1} + \left\lceil \frac{m+1}{a_2} \right\rceil,$$

$$k_{2j+1} + \left\lceil \frac{n}{b_2} \right\rceil \leq k < k_{2j+1} + \left\lceil \frac{n+1}{b_2} \right\rceil.$$

Then m, n satisfy the following relationship

$$\frac{b_2}{a_2}m - b_2 - 1 \leq n \leq \frac{b_2}{a_2}m + \frac{b_2}{a_2} + b_2. \quad (29)$$

Like the estimate (23) for $d_k(S)$, we have for $d_k(T)$

$$\widetilde{f}_j^*(m, n) \leq d_k(T) \leq \widetilde{g}_j^*(m, n),$$

where

$$\widetilde{f}_j^*(m, n) = a_2 \frac{b_1(k_{2j+1} - k_{2j}) + n}{a_2 k_{2j+1} + 1 + m}$$

and

$$\widetilde{g}_j^*(m, n) = a_2 \frac{k_{2j} + b_1(k_{2j+1} - k_{2j}) + 1 + n}{a_2 k_{2j+1} - a_2 + m}.$$

This estimate of $d_k(T)$ together with (23) gives

$$\widetilde{f}_j(m) + \widetilde{f}_j^*(m, n) \leq d_k(S) + d_k(T) \leq \widetilde{g}_j(m) + \widetilde{g}_j^*(m, n).$$

which, combined with (29), yields

$$\widetilde{F}_j(m) \leq d_k(S) + d_k(T) \leq \widetilde{G}_j(m), \quad (30)$$

where

$$\widetilde{F}_j(m) = (a_2 + b_2) \frac{\frac{a_2(a_1+b_1)}{a_2+b_2}(k_{2j+1} - k_{2j}) - \frac{a_2+a_2b_2}{a_2+b_2} + m}{a_2 k_{2j+1} + 1 + m}$$

and

$$\widetilde{G}_j(m) = (a_2 + b_2) \frac{\frac{2a_2}{a_2+b_2}k_{2j} + \frac{a_2(a_1+b_1)}{a_2+b_2}(k_{2j+1} - k_{2j}) + \frac{b_2+a_2b_2+2a_2}{a_2+b_2} + m}{a_2 k_{2j+1} - a_1 + m}.$$

For sufficiently large k we have from (30)

- when $a_1 + b_1 < a_2 + b_2$, both $\widetilde{F}_j(m)$ and $\widetilde{G}_j(m)$ are increasing as m goes from 0 to $M_{2j+2} - 1$, so $\widetilde{F}_j(0) \leq d_k(S) + d_k(T) \leq \widetilde{G}_j(M_{2j+2} - 1)$;
- when $a_1 + b_1 = a_2 + b_2$, $\widetilde{F}_j(m)$ is increasing and $\widetilde{G}_j(m)$ is decreasing as m goes from 0 to $M_{2j+2} - 1$, so $\widetilde{F}_j(0) \leq d_k(S) + d_k(T) \leq \widetilde{G}_j(0)$;
- when $a_1 + b_1 > a_2 + b_2$, both $\widetilde{F}_j(m)$ and $\widetilde{G}_j(m)$ are decreasing as m goes from 0 to $M_{2j+2} - 1$, so $\widetilde{F}_j(M_{2j+2} - 1) \leq d_k(S) + d_k(T) \leq \widetilde{G}_j(0)$.

Noting from (14) and the definition of M_{2j+i} that

$$\lim_{j \rightarrow \infty} F_j(M_{2j+1} - 1) = \lim_{j \rightarrow \infty} G_j(M_{2j+1} - 1) = \lim_{j \rightarrow \infty} \widetilde{F}_j(0) = \lim_{j \rightarrow \infty} \widetilde{G}_j(0) = a_1 + b_1$$

and

$$\lim_{j \rightarrow \infty} F_j(0) = \lim_{j \rightarrow \infty} G_j(0) = \lim_{j \rightarrow \infty} \widetilde{F}_j(M_{2j+2} - 1) = \lim_{j \rightarrow \infty} \widetilde{G}_j(M_{2j+2} - 1) = a_2 + b_2,$$

we get from the above estimates of $d_k(S) + d_k(T)$

$$\liminf_{k \rightarrow \infty} (d_k(S) + d_k(T)) \geq \min\{a_1 + b_1, a_2 + b_2\}$$

and

$$\limsup_{k \rightarrow \infty} (d_k(S) + d_k(T)) \leq \max\{a_1 + b_1, a_2 + b_2\},$$

which, combined with (26), give the desired equalities. \square

Remark 1. Lemma 3 and Lemma 4 can be applied to provide examples of sets with $\dim_H E^d > d \dim_H E$. Indeed, let $S = S(\{k_n\}_{n \geq 0}, 1/2, 1/4)$ and $T = S(\{k_n\}_{n \geq 0}, 1/4, 1/3)$. Let E_S and E_T be the corresponding subset of $[0, 1]$ defined by (5). Then we have $\dim_H E_S = \dim_H E_T = 1/4$ by Lemma 3 and $\dim_H(E_S \times E_T) = 7/12$ by Lemma 4. Take $E = E_S \cup E_T$. One has $\dim_H E = 1/4$ and $\dim_H(E \times E) \geq \dim_H(E_S \times E_T) = 7/12 > 2 \dim_H E$.

Lemma 5. Let E and F be metric spaces and d a positive integer. Then

$$\dim_H(E^d \times F^d) = \dim_H(E \times F)^d.$$

A similar equality holds for both the packing dimension and the upper box-counting dimension.

Proof. As mentioned, the product $X := \prod_{i=1}^d (X_i, \rho_i)$ of metric spaces (X_i, ρ_i) has been equipped with the metric

$$\rho(x, y) = \left(\sum_{i=1}^d (\rho_i(x_i, y_i))^2 \right)^{\frac{1}{2}}, \quad x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in X.$$

It is easy to see that $E^d \times F^d$ is isometric to $(E \times F)^d$, so the desired equality follows. \square

Proof of Theorem 1. Let $\alpha, \beta, \gamma, \lambda$ be positive with $\beta \leq \gamma$ and $\alpha + \beta \leq \lambda \leq \alpha + \gamma$. Let d be an integer such that $\frac{\alpha}{d}, \frac{\beta}{d}, \frac{\gamma}{d}, \frac{\lambda}{d} \in (0, 1)$. Let $S = S(\{k_n\}_{n \geq 0}, \frac{\lambda - \beta}{d}, \frac{\alpha}{d})$ and $T = S(\{k_n\}_{n \geq 0}, \frac{\beta}{d}, \frac{\gamma}{d})$ be subsets of \mathbb{N} defined by (16). Let E_S and E_T be the corresponding subset of $[0, 1]$ defined by (5). By Lemmas 3 and 4,

$$\dim_H E_S = \frac{\alpha}{d}, \quad \dim_H E_T = \frac{\beta}{d}, \quad \dim_P E_T = \frac{\gamma}{d}, \quad \dim_H(E_S \times E_T) = \frac{\lambda}{d},$$

which, together with Lemmas 1, 2, and 5, yields

$$\dim_H E_S^d = \alpha, \quad \dim_H E_T^d = \beta, \quad \dim_P E_T^d = \gamma, \quad \dim_H(E_S^d \times E_T^d) = \lambda.$$

This proves Theorem 1 by taking $E = E_S^d$ and $F = E_T^d$. \square

Proof of Theorem 2. Let $\alpha, \beta, \gamma, \lambda$ be positive with $\alpha \leq \gamma$ and $\alpha + \beta \leq \lambda \leq \gamma + \beta$. Let d be an integer such that $\frac{\alpha}{d}, \frac{\beta}{d}, \frac{\gamma}{d}, \frac{\lambda}{d} \in (0, 1)$. Let $S = S(\{k_n\}_{n \geq 0}, \frac{\alpha}{d}, \frac{\gamma}{d})$

and $T = S(\{k_n\}_{n \geq 0}, \frac{\beta}{d}, \frac{\lambda - \gamma}{d})$ defined by (16). Let $E = E_S^d$ and $F = E_T^d$. Then, by the results in Section 2 and Lemma 5, one has

$$\dim_H E = \alpha, \dim_P F = \beta, \dim_P E = \gamma, \dim_P(E \times F) = \lambda.$$

This proves Theorem 2. \square

Proof of Theorem 3. Let $\alpha, \beta, \gamma, \lambda$ be positive with $\alpha \leq \gamma$ and $\alpha + \beta \leq \lambda \leq \gamma + \beta$. For the sets E and F arising in the proof of Theorem 2, one has

$$\underline{\dim}_B E = \alpha, \overline{\dim}_B F = \beta, \overline{\dim}_B E = \gamma, \overline{\dim}_B(E \times F) = \lambda.$$

This proves Theorem 3. \square

References

- [1] Besicovitch A S, Moran P A P. The measure of product and cylinder sets. J London Math Soc, 1945; 20: 110-120.
- [2] Billingsley P. Ergodic Theory and Information. John Wiley & Sons, 1965.
- [3] Bishop C J, Peres Y. Fractal Sets in Probability and Analysis. Preprint. <http://www.math.sunysb.edu/~bishop/all2.pdf>
- [4] Falconer K J. Fractal Geometry-Mathematical Foundations and Applications. John Wiley & Sons, 1990.
- [5] Howroyd J D. On Hausdorff and packing dimension of product spaces. Math Proc Camb Phil Soc, 1996; 119: 715-727.
- [6] Luukkainen J. Assouad dimension: Antifractal metrization, porous sets, and homogeneous measures. J Korean Math Soc, 1998; 35: 23-76.
- [7] Marstrand J M. The dimension of the Cartesian product sets. Proc Cambridge Philos Soc, 1954; 50: 198-202.
- [8] Mattila P. Geometry of Sets and Measures in Euclidean Spaces. Cambridge University Press, 1995.
- [9] Peng F J, Wang W, Wen S Y. On Assouad dimension of products. Preprint.
- [10] Tricot C. Two definitions of fractional dimension. Math Proc Cambridge Philos Soc, 1982; 91: 57-74.
- [11] Wen Z Y. Mathematical Foundations of Fractal Geometry. Shanghai Scientific and Technological Education Publishing House, SHANGHAI, 2000.

Chun WEI and Zhixiong WEN
 Department of Mathematics,
 Huazhong University of Science and Technology,

Wuhan 430074, China

Shengyou WEN
Department of Mathematics,
Hubei University,
Wuhan 430062, China